Interacting Wess-Zumino-Novikov-Witten Models

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Abstract

We study the system of two WZNW models coupled to each other via the current-current interaction. The system is proven to possess the strong/weak coupling duality symmetry. The strong coupling phase of this theory is discussed in detail. It is shown that in this phase the interacting WZNW models approach non-trivial conformal points along the renormalization group flow. The relation between the principal chiral model and interacting WZNW models is investigated.

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1 Introduction

String theory symmetries act on the space of string solutions that from the point of view of the world-sheet description can be observed as transformations of the coupling constants of underlying two-dimensional quantum field theories. Therefore, some information about local properties of these symmetries can be extracted by infinitesimal variations of the given string solution. In other words, one can make use of perturbation theory methods in order to study the string symmetries. It may seem that the so-called duality symmetries of string theory will fail to be detected within this perturbative approach. However, in the vicinity of the self-dual points one can still carry on with a proper perturbative expansion [1]. Of course, the problem now will be how to identify the self-dual points? Apart from the last issue, perturbation theory appears to be an adequate tool in elaborating string symmetries.

Some of the perturbations of a given two-dimensional conformal field theory corresponding to a certain string solution may take us away from the space of string solutions. This will happen when the perturbation breaks the conformal invariance. Therefore, in order to reveal some of the string symmetries, it may be necessary to extend the space of string solutions. Then, restricted to string solutions, these symmetries either will be seen as discrete or will not be seen at all.

The hope is that by integrating infinitesimal transformations it will be possible to reconstruct the entire group of string symmetries. Correspondingly, this may turn out to bring us to a better understanding of string field theory itself. Luckily, one may find a two-dimensional model which corresponds to a general element of the entire string symmetry group or its subgroup. In the present paper we shall argue that such a general element can be described as a system of two WZNW models coupled to each other through a current-current Thirring-like interaction. As yet we have not proven this conjecture, but we have found some supporting evidence for it.

The importance of interacting WZNW models also arises as one tries to find a unified description of two-dimensional integrable models. The latter have proven to be extremely useful in modeling many physical situations and phenomena. At present it seems that

these models may be of great importance in understanding non-trivial properties of string theory, for example, S-duality [2]. So far only the principal chiral model has been considered in this context [2]. In spite of its merits, this model, in many cases, appears to be an approximation, which does not always allow one to grasp some of the essential effects of a system it is supposed to describe. For example, the hidden affine symmetry of the non-linear sigma model [3] is not sufficient for describing the conformal Sugawara construction, whose proper description is achieved in terms of the Wess-Zumino-Novikov-Witten (WZNW) model [4]. One can observe the latter as a modified PCM [5]. However, it seems to be more appropriate to regard the PCM as a modified WZNW theory [6].

The way in which the PCM stems from the WZNW model suggests that there may exist a larger class of two-dimensional field theories which possess the important property of integrability. We shall discuss this relation between the PCM and the WZNW model in section 3. It turns out that a broad class of two-dimensional integrable theories can be described in terms of interacting WZNW models. The PCM corresponds to the isoscalar current-current interaction [6]. More general current-current interactions give rise to more general integrable models. Our hope is that the space of the given integrable models may form a connected multitude parametrized by the Thirring coupling constants. This space of theories might turn out to be of help in understanding the space of string compactifications and solutions, which ought to be presented by CFT's in this space. The geometry and topology of this model space may play a role in formulating the background independent string field action.

This paper is organized as follows. In section 2, two interacting level k_1 and k_2 WZNW models are introduced and their classical properties are studied. In section 3, the PCM is presented as a system of two interacting WZNW models. In section 4, we exhibit the strong/weak coupling duality of interacting WZNW models at the quantum level. In section 5, by using this duality, we analyze the strong coupling phase of the theory in question. We show that the theory has a non-trivial critical point in this phase. In section 6, we discuss the non-perturbative conformal points of the system of two interacting WZNW models. We conclude in section 7.

2 Classical properties of interacting WZNW models

We would like to start with some classical properties of interacting WZNW models, which then will be of help in developing quantum theory.

Let $S_{WZNW}(g_1, k_1)$ and $S_{WZNW}(g_2, k_2)$ be the actions of two WZNW models of level k_1 and k_2, and with g_1 and g_2 taking values in the Lie groups G_1 and G_2 respectively:

$$S_{WZNW}(g_1, k_1) = \frac{-k_1}{4\pi} \left\{ \int \mathrm{Tr}|g_1^{-1} \, \mathrm{d}g_1|^2 + \frac{i}{3} \int \mathrm{d}^{-1}\mathrm{Tr}(g_1^{-1}\mathrm{d}g_1)^3 \right\},$$

$$\tag{2.1}$$

$$S_{WZNW}(g_2, k_2) = \frac{-k_2}{4\pi} \left\{ \int \mathrm{Tr}|g_2^{-1} \, \mathrm{d}g_2|^2 + \frac{i}{3} \int \mathrm{d}^{-1}\mathrm{Tr}(g_2^{-1}\mathrm{d}g_2)^3 \right\}.$$

These models are invariant under the following affine symmetries

$$g_1 \rightarrow \bar{\Omega}_1(\bar{z})g_1\Omega_1(z),$$

$$\tag{2.2}$$

$$g_2 \rightarrow \bar{\Omega}_2(\bar{z})g_2\Omega_2(z),$$

where the parameters $\Omega_{1,2}$, $\bar{\Omega}_{1,2}$ are arbitrary independent functions of z and $\bar{z}$ respectively.

It turns out that there is an interaction between these two WZNW models which preserves the affine symmetries of the free theories. This interaction is given as follows

$$S_I = \frac{-k_1 k_2}{\pi} \int d^2 z \mathrm{Tr}^2(g_1^{-1}\partial g_1 \ S \ \bar{\partial}g_2 g_2^{-1}), \tag{2.3}$$

with the coupling S belonging to the direct product of two Lie algebras $\mathcal{G}_1 \otimes \mathcal{G}_2$. The symbol Tr^2 indicates a double tracing over the indices of a matrix from the tensor product $\mathcal{G}_1 \otimes \mathcal{G}_2$. We shall assume that S is invertible, so that $\dim G_1 = \dim G_2$.

We call the system described by the following action

$$S(g_1, g_2, S) = S_{WZNW}(g_1, k_1) + S_{WZNW}(g_2, k_2) + S_I(g_1, g_2, S) \tag{2.4}$$

a system of two interacting WZNW models. Remarkably, when $S \neq 0$, the interacting system still has $\hat{G}_1^L \otimes \hat{G}_1^R \otimes \hat{G}_2^L \otimes \hat{G}_2^R$ affine symmetry [7], under which

$$g_1 \rightarrow \bar{\Omega}_1(\bar{z})g_1 h_1 \Omega_1(z)h_1^{-1},$$

$$\tag{2.5}$$

$$g_2 \rightarrow h_2^{-1}\bar{\Omega}_2(\bar{z})h_2 g_2 \Omega_2(z),$$

where h_1, h_2 are non-local functions of g_1, g_2 satisfying

$$\bar{\partial} h_1 h_1^{-1} = 2k_2 \text{Tr} S \bar{\partial} g_2 g_2^{-1},$$

$$(2.6)$$

$$h_2^{-1} \partial h_2 = 2k_1 \text{Tr} S g_1^{-1} \partial g_1.$$

The $\bar{\Omega}_1$ and Ω_2 transformations remain local, while the $\bar{\Omega}_2$ and Ω_1 transformations are now intrinsically non-local, as they involve h_1, h_2. The local $\bar{\Omega}_1$ and Ω_2 symmetries are manifest, whereas the proof of the remaining non-local ones is given in [8].

The fact that the interacting theory possesses the affine symmetry amounts to the existence of an infinite number of affine currents and, as a consequence of it, to the integrability of the theory. Like in the ordinary WZNW model, the affine currents of interacting WZNW models emerge via the equations of motion. Indeed, the equations of motion of the theory in eq. (2.4) can be written as

$$\partial \bar{\mathcal{J}}_{(1)} = 0, \qquad \bar{\partial} \mathcal{J}_{(2)} = 0, \qquad (2.7)$$

where

$$\bar{\mathcal{J}}_{(1)}^{\bar{a}} = \bar{J}_{(1)}^{\bar{a}} + 2k_2 \phi_1^{a\bar{a}} S^{a\bar{b}} \bar{J}_{(2)}^{\bar{b}},$$

$$(2.8)$$

$$\mathcal{J}_{(2)}^{a} = J_{(2)}^{a} + 2k_1 \phi_2^{a\bar{a}} S^{b\bar{a}} J_{(1)}^{b},$$

with

$$J_{(1)}^{a} = -\frac{k_1}{2} \text{Tr}(g_1^{-1} \partial g_1 \, t_1^a),$$

$$J_{(2)}^{a} = -\frac{k_2}{2} \text{Tr}(g_2^{-1} \partial g_2 \, t_2^a),$$

$$\bar{J}_{(1)}^{\bar{a}} = -\frac{k_1}{2} \text{Tr}(\bar{\partial} g_1 g_1^{-1} \, t_1^{\bar{a}}),$$

$$(2.9)$$

$$\bar{J}_{(2)}^{\bar{a}} = -\frac{k_2}{2} \text{Tr}(\bar{\partial} g_2 g_2^{-1} \, t_2^{\bar{a}}),$$

$$\phi_1^{a\bar{a}} = \text{Tr}(g_1 t_1^a g_1^{-1} t_1^{\bar{a}}),$$

$$\phi_2^{a\bar{a}} = \text{Tr}(g_2 t_2^a g_2^{-1} t_2^{\bar{a}}),$$

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where $t^i_{1,2}$ are the generators of the Lie algebras $\mathcal{G}_{1,2}$ associated with the Lie groups $G_{1,2}$,

$$\left[t^i_1, t^j_1\right] = f^{ij}_{(1)k} t^k_1,$$

$$\left[t^i_2, t^j_2\right] = f^{ij}_{(2)k} t^k_2,$$

(2.10)

with $f^{ij}_{(1,2)k}$ the structure constants.

The analytical currents $\bar{\mathcal{J}}_{(1)}$, $\mathcal{J}_{(2)}$ correspond to the two local $\bar{\Omega}_1$ and Ω_2 symmetries respectively. There are in addition two non-local conserved currents $\mathcal{J}_{(1)}$, $\bar{\mathcal{J}}_{(2)}$ corresponding to the two non-local $\bar{\Omega}_2$ and Ω_1 symmetries. These two local and two non-local conserved currents generate an infinite number of conserved charges that results in the integrability of the system of two interacting WZNW models with *arbitrary* coupling matrix S. For the local currents $\bar{\mathcal{J}}_{(1)}$, $\mathcal{J}_{(2)}$ an infinite set of conserved charges can be built up according to

$$\bar{\mathcal{J}}_{(1)n} = \oint \frac{d\bar{z}}{2\pi i}\, \bar{z}^n \bar{\mathcal{J}}_{(1)}(\bar{z}),$$

$$\mathcal{J}_{(2)n} = \oint \frac{dz}{2\pi i}\, z^n \mathcal{J}_{(2)}(z).$$

(2.11)

With respect to the classical Poisson bracket the charges $\bar{\mathcal{J}}_{(1)n}$, $\mathcal{J}_{(2)n}$ form the two affine algebras $\hat{\bar{\mathcal{G}}}_1$ and $\hat{\mathcal{G}}_2$ at levels k_1 and k_2, respectively,

$$\left[\bar{\mathcal{J}}^{\bar{a}}_{(1)n}, \bar{\mathcal{J}}^{\bar{b}}_{(1)m}\right] = f^{\bar{a}\bar{b}}_{(1)\bar{c}}\, \bar{\mathcal{J}}^{\bar{c}}_{(1)n+m} + \frac{k_1}{2}\delta^{\bar{a}\bar{b}} m \delta_{n+m,0},$$

$$\left[\mathcal{J}^a_{(2)n}, \mathcal{J}^b_{(2)m}\right] = f^{ab}_{(2)c}\, \mathcal{J}^c_{(2)n+m} + \frac{k_2}{2}\delta^{ab} m \delta_{n+m,0}.$$

(2.12)

The point to be made is that the existence of the analytical currents $\bar{\mathcal{J}}_{(1)}$, $\mathcal{J}_{(2)}$ follows directly from the equations of motion. Therefore, in quantum theory, these currents will exist as long as the equations of motion can be promoted to the quantum level. Hence, whenever the system of interacting WZNW models exists as a quantum theory, it will be automatically integrable.

There is another interesting classical property of interacting WZNW models. Namely, given a system of two interacting WZNW models one can find a dual theory [6]. The

procedure is as follows. First, we introduce auxiliary Lie-algebra-valued variables Q, $\bar{P}$ and write the interaction action in the (first-order) form

$$S(g_1, g_2, Q, \bar{P}) \;\; = \;\; S_{WZNW}(g_1, k_1) \; + \; S_{WZNW}(g_2, k_2)$$

$$(2.13)$$

$$- \;\; \frac{1}{2\pi} \int d^2 z \, \mathrm{Tr}(k_1 g_1^{-1} \partial g_1 \bar{P} \; + \; k_2 \bar{Q} \bar{\partial} g_2 g_2^{-1} \; - \; \frac{1}{2} Q \cdot S^{-1} \cdot \bar{P}).$$

The requirement of invertibility of S comes from the last equation. If one eliminates Q, $\bar{P}$ via their equations of motion, one regains (2.4). Instead, we express Q, $\bar{P}$ in terms of new variables h_1, h_2 as

$$\bar{\partial} h_1 h_1^{-1} = \bar{P}, \qquad h_2^{-1} \partial h_2 = Q \qquad (2.14)$$

and use the Polyakov-Wiegmann formula [9] to obtain

$$S(g_1, g_2, Q, \bar{P}) \;\; = \;\; S_{WZNW}(g_1 h_1, k_1) \; + \; S_{WZNW}(h_2 g_2, k_2) \; + \; S_{WZNW}(h_1, -k_1)$$

$$(2.15)$$

$$+ \;\; S_{WZNW}(h_2, -k_2) \; + \; \frac{1}{4\pi} \int d^2 z \, \mathrm{Tr}^2 h_2^{-1} \partial h_2 S^{-1} \bar{\partial} h_1 h_1^{-1}.$$

If we introduce new variables

$$\tilde{g}_1 = g_1 h_1, \qquad \tilde{g}_2 = h_2 g_2, \qquad (2.16)$$

then we can see that $\tilde{g}_1$, $\tilde{g}_2$ decouple from h_1, h_2 and the action for the latter is the interacting system of two level $-k_1$, $-k_2$ WZNW models with the coupling

$$\tilde{S} = -\frac{1}{4k_1 k_2} S^{-1}. \qquad (2.17)$$

At the quantum level the given duality undergoes some quantum modifications which will be the subject of section 4.

We have exhibited that the system of interacting WZNW models is integrable and possesses the duality symmetry. In our further discussion we shall study these classical properties for quantum theory.

3 The PCM from WZNW models

For the further purposes it is instructive to demonstrate a curious relation between the PCM and the interacting system of WZNW models we have described in the previous section.

Let us consider the system of two interacting identical WZNW models with the isoscalar coupling matrix

$$S = s\, I, \tag{3.18}$$

where I is the identity in $\mathcal{G} \otimes \mathcal{G}$, s is some parameter. In this case, the affine currents in eqs. (2.8) take the form

$$\bar{\mathcal{J}}_1 = \bar{J}_1 \;+\; 2ksg_1\bar{J}_2g_1^{-1},$$

$$\tag{3.19}$$

$$\mathcal{J}_2 = J_2 \;+\; 2ksg_2^{-1}J_1g_2.$$

Correspondingly the equations of motion (2.7) can be rearranged as follows

$$\bar{\partial}J_1 \;-\; \frac{4s}{1+2ks}\left[J_1, \bar{J}_2\right] \;=\; 0,$$

$$\tag{3.20}$$

$$\partial\bar{J}_2 \;-\; \frac{4s}{1+2ks}\left[J_1, \bar{J}_2\right] \;=\; 0.$$

It is convenient to introduce new variables

$$A = J_1, \qquad \bar{A} = -\bar{J}_2 \tag{3.21}$$

to rewrite eqs. (3.20) in the form

$$\bar{\partial}A \;+\; \lambda\left[A, \bar{A}\right] \;=\; 0,$$

$$\tag{3.22}$$

$$\partial\bar{A} \;-\; \lambda\left[A, \bar{A}\right] \;=\; 0,$$

where

$$\lambda = \frac{4s}{1+2ks}. \tag{3.23}$$

The given equations are nothing but the equations of motion of the PCM (see e.g. [2]). Thus, we have exhibited the classical equivalence between the latter and the system of two interacting WZNW models. In particular, this equivalence allows us to apply the results in Ref.[2] to the interacting WZNW models. It might be interesting to connect the affine symmetries in eqs. (2.5) with the hidden affine symmetries of the PCM [2].

The PCM was shown to correspond to the isoscalar interaction between two identical WZNW theories. In general the system of two interacting WZNW models provides a uniform description to a large class of integrable theories some of which might turn out to be new.

4   The quantum duality

Because duality symmetries occur to play an important role in physics, we shall discuss in detail the duality symmetry of the system of two interacting WZNW models at the quantum level. Later on we shall make use of this symmetry to solve the theory in the strong coupling regime.

Let us consider the partition function of the system of two interacting WZNW models

$$Z(k_1, k_2, S) = \int \mathcal{D}g_1 \mathcal{D}g_2 \exp[-S(g_1, g_2, S)]. \tag{4.24}$$

It is necessary to point out that the definition of the function integral in eq. (4.24) in general depends on the coupling constant matrix S. For example, there is a critical point (Polyakov-Wiegmann conformal point) at which the system of interacting WZNW models acquires the gauge symmetry. Therefore, the functional integral at this point has to be further defined in the Faddeev-Popov manner.

As in the classical case, we introduce the auxiliary variables under the functional integral to obtain

$$Z(k_1, k_2, S) = J^{-1} \int \mathcal{D}g_1 \mathcal{D}g_2 \mathcal{D}\bar{P} \mathcal{D}Q \exp[-S(g_1, g_2, Q, \bar{P})],$$

$$\tag{4.25}$$

$$J = \int \mathcal{D}\bar{P} \mathcal{D}Q \exp\left[-\frac{1}{4\pi} \int d^2x \mathrm{Tr} \bar{P} S^{-1} Q\right].$$

Here the quantity J depends only on topology of the world sheet. Integration over Q, $\bar{P}$ in the partition function Z amounts to using the equations of motion for Q, $\bar{P}$ giving rise to the equivalence of the two expressions in (4.24) and (4.25). The step further from the variables Q, $\bar{P}$ to the variables h_1, h_2 entails some new features compared to the classical theory. Namely, this change of variables is accompanied by the Jacobians and the local counterterms which affect the interaction between h_1 and h_2. All of these modifications can be properly taken into account via the following formula

$$Z(k_1, k_2, S) = J^{-1} Z^{(1)}_{WZNW}(k_1) Z^{(2)}_{WZNW}(k_2) \bar{Z}^{(1)}_{ghost} Z^{(2)}_{ghost}$$

$$\times \ Z\left(-k_2 - 2c_V(G_2), -k_1 - 2c_V(G_1), -\frac{A(k_1, k_2)S^{-1}}{4(k_1 + 2c_V(G_1))(k_2 + 2c_V(G_2))}\right),$$

(4.26)

where $Z^{(1,2)}_{WZNW}(k_{1,2})$ are the partition functions of the WZNW models on $G_{1,2}$ with levels $k_{1,2}$, whereas

$$\bar{Z}^{(1)}_{ghost} = \int \mathcal{D}b \mathcal{D}c \exp\left[-\int d^2 z \mathrm{Tr}_1 b\bar{\partial}c\right],$$

$$Z^{(2)}_{ghost} = \int \mathcal{D}\bar{b}\mathcal{D}\bar{c} \exp\left[-\int d^2 z \mathrm{Tr}_2 \bar{b}\partial\bar{c}\right],$$

(4.27)

with b, c, $\bar{b}$, $\bar{c}$ being Lie-algebra-valued ghost-like fields arising in the course of the change of variables in eqs. (2.14). The constant $A(k_1, k_2)$ stands for the corrections due to the local counterterms. We shall fix A from the consistency condition.

Note that the appearance of the ghosts is fairly natural because the dual theory involves WZNW models at negative level. Therefore, the unitarity of the whole system assumes making use of some BRST-like procedure for ruling out negative normed states. We shall discuss the unitarity of the model under consideration in some particular cases.

Let us turn to the consistency condition on the constant A. It comes along in the following way. By applying the duality transformation repeatedly to the dual theory, one can derive the following identity

$$Z(k_1, k_2, S) = J^{-2} Z^{(1)}_{WZNW}(k_1) Z^{(2)}_{WZNW}(k_2) \bar{Z}^{(1)}_{ghost} Z^{(2)}_{ghost}$$

$$\times \ Z^{(2)}_{WZNW}(k_2 - 2c_V(G_2)) Z^{(1)}_{WZNW}(k_1 - 2c_V(G_1)) \bar{Z}^{(2)}_{ghost} Z^{(1)}_{ghost}$$

(4.28)

$$\times \quad Z\left(k_1, k_2, \frac{A(-k_1 - 2c_V(G_1), -k_2 - 2c_V(G_2))(k_1 + 2c_V(G_1))(k_2 + 2c_V(G_2))}{A(k_1, k_2)k_1 k_2} S\right).$$

The crucial observation is that

$$Z_{WZNW}^{(1)}(k_1) \quad Z_{WZNW}^{(2)}(k_2) \, Z_{WZNW}^{(1)}(k_1 - 2c_V(G_1)) \, Z_{WZNW}^{(2)}(k_2 - 2c_V(G_2))$$

$$(4.29)$$

$$\times \qquad \bar{Z}_{ghost}^{(1)} \, Z_{ghost}^{(1)} \, \bar{Z}_{ghost}^{(2)} \, Z_{ghost}^{(2)} = Z_{G_1/G_1} \, Z_{G_2/G_2},$$

where $Z_{G_{1,2}/G_{1,2}}$ are the partition functions of the topological conformal field theories [10]. Thus, formula (4.28) takes the form

$$Z(k_1, k_2, S) = J^{-2} \, Z_{G_1/G_1} \, Z_{G_2/G_2} \, Z(k_1, k_2, S'), \qquad (4.30)$$

where

$$S' = \frac{(k_1 + 2c_V(G_1))(k_2 + 2c_V(G_2))A(-k_1 - 2c_V(G_1), -k_2 - 2c_V(G_2))}{k_1 k_2 A(k_1, k_2)} S. \qquad (4.31)$$

We have shown in [8] that J can be expressed as follows

$$J^2 = Z_{G_1/G_1} \, Z_{G_2/G_2}. \qquad (4.32)$$

Thus, the factor J^{-2} cancels with the factor $Z_{G_1/G_1} \, Z_{G_2/G_2}$ and we arrive at the following equality

$$Z(k_1, k_2, S) = Z(k_1, k_2, S'). \qquad (4.33)$$

The last formula implies that

$$S = S' \qquad (4.34)$$

up to modular transformations in the space of coupling constants. All in all we find the consistency condition

$$\frac{A(-k_1 - 2c_V(G_1), -k_2 - 2c_V(G_2))}{A(k_1, k_2)} = \frac{k_1 k_2}{(k_1 + 2c_V(G_1))(k_2 + 2c_V(G_2))}. \qquad (4.35)$$

Apart from the modular symmetry, this consistency condition may have many solutions. To pick up the right one, one has to consider the classical limit. Taking into account eq. (2.15), it is not difficult to see that the classical limit amounts to the following condition

$$\lim_{k_1, k_2 \to \infty} A(k_1, k_2) = 1. \qquad (4.36)$$

Bearing in mind this limit the constant A is fixed without ambiguities

$$A(k_1, k_2) = \left[\frac{(k_1 + 2c_V(G_1))(k_2 + 2c_V(G_2))}{k_1 k_2} \right]^{1/2}. \tag{4.37}$$

Finally we write done the exact formula for the duality symmetry of the system of two interacting WZNW models:

$$Z(k_1, k_2, S) = J^{-1} Z^{(1)}_{WZNW}(k_1) Z^{(2)}_{WZNW}(k_2) \bar{Z}^{(1)}_{ghost} Z^{(2)}_{ghost}$$

$$\tag{4.38}$$

$$\times \ Z\left(-k_2 - 2c_V(G_2), -k_1 - 2c_V(G_1), -\frac{S^{-1}}{4\sqrt{(k_1 + 2c_V(G_1))(k_2 + 2c_V(G_2))k_1 k_2}} \right).$$

Nicely there is a check for eq. (4.38). Let us consider the case, when $k_1 = k_2 = k$, $G_1 = G_2 = G$. The system of two identical interacting WZNW models has the Polyakov-Wiegmann conformal point at

$$S_{PW} = \frac{I}{2k}, \tag{4.39}$$

where I is the identity from $\mathcal{G} \otimes \mathcal{G}$. At this point eq. (3.38) reads off

$$Z\left(k, k, \frac{I}{2k} \right) = J^{-1} Z^2_{WZNW}(k) \bar{Z}_{ghost} Z_{ghost}$$

$$\tag{4.40}$$

$$\times \ Z\left(-k - 2c_V(G), -k - 2c_V(G), -\frac{I}{2(k + 2c_V(G))} \right).$$

Now one can see that the PW conformal point of the original theory goes to the PW conformal point of the dual theory. This means that the both theories possess the gauge symmetry. Thus, we have to impose an appropriate gauge condition in order to fix the given gauge arbitrariness. This can be done without introducing dynamical Faddeev-Popov ghosts by setting one of the two group elements to the identity matrix. After that one can evaluate the Virasoro central charge of the original CFT and the dual one. Based on it, one can verify that equality (4.38) persists for the Virasoro central charges.

Note that the dual theory in eq. (4.38) describes the interaction between two WZNW models with negative levels. Because of that, this theory does not belong to the space of interacting WZNW models with positive levels. Thus, the duality transformation links

the two theories of different types. Remarkably, the strong coupling phase of one theory goes into the weak coupling phase of its dual one. In the next section we shall show how to make use of the given duality in exploring the strong coupling regime of the system of two interacting WZNW models with positive levels.

5   The strong coupling phase

Somewhat to simplify the further consideration we put $k_1 = k_2 = k$, $G_1 = G_2 = G$. The formulation of the system of two interacting WZNW models as given in section 2, turns out to be convenient in analysis of the weak coupling phase. Indeed, in this regime one can expand around the free WZNW models in the interaction, which is determined by a marginal conformal operator. However, when the coupling matrix S is large, one can no longer relay on perturbation in S. Our aim in this section is to reformulate the theory in eq. (2.4) so that it will be suitable for investigation of the strong coupling phase.

The idea is to make the following change of variables

$$g_1 \to h(\tilde{g}_1) \cdot \tilde{g}_2, \qquad g_2 \to \tilde{g}_1, \tag{5.41}$$

where $\tilde{g}_1$, $\tilde{g}_2$ are new variables, whereas the function $h(\tilde{g}_1)$ is the solution of the following equation

$$\partial h h^{-1} = -2k \text{Tr} S \tilde{g}_1^{-1} \partial \tilde{g}_1. \tag{5.42}$$

This determines h up to changes of the form

$$h \to h \Lambda(\bar{z}), \tag{5.43}$$

where Λ is an antiholomorphic matrix function, $\partial \Lambda = 0$, and we will pick some particular solution $h_0(z, \bar{z})$. One can take any other solution $h_0 \Lambda$. However, because the Jacobian of the change in eq. (5.41) is always equal to one, a new choice will not change the theory.

By considering $\bar{\partial}(\partial h h^{-1})$ and using eq. (5.42) we also obtain

$$\bar{\partial} h_0 h_0^{-1}(z, \bar{z}) = -2k \text{Tr} S \tilde{g}_1^{-1} \bar{\partial}_{\bar{z}} \tilde{g}_1(z, \bar{z})$$

$$\tag{5.44}$$

$$+ \ 2k \int d^2 z \bar{\partial}_{\bar{z}} G(z, \bar{z}, y, \bar{y}) v(y, \bar{y}),$$

where

$$v(y, \bar{y}) \; = \; \text{Tr} S \tilde{g}_1^{-1}(y, \bar{y}) \left[\bar{\partial}_{\bar{y}} \tilde{g}_1(y, \bar{y}) \tilde{g}_1^{-1}, \; \partial_y \tilde{g}_1(y, \bar{y}) \tilde{g}_1^{-1}(y, \bar{y}) \right] \tilde{g}_1(y, \bar{y})$$

$$(5.45)$$

$$+ \; \left[\text{Tr} S \tilde{g}_1^{-1}(y, \bar{y}) \partial_y \tilde{g}_1(y, \bar{y}), \; \bar{\partial} h_0(y, \bar{y}) h_0^{-1}(y, \bar{y}) \right]$$

and the Green function $G(z, \bar{z}; y, \bar{y})$ satisfies

$$\bar{\partial}_{\bar{z}} \partial_z G(z, \bar{z}; y, \bar{y}) = \delta(z, y) \delta(\bar{z}, \bar{y}). \qquad (5.46)$$

We regularize the Green function in such a way that

$$\lim_{y \to z, \bar{y} \to \bar{z}} \partial_z G(z, \bar{z}; y, \bar{y}) = 0 \qquad (5.47)$$

so that, despite its nonlocality, the right hand side of eq. (5.44) is well defined, even when $(y, \bar{y}) \to (z, \bar{z})$.

Equations (5.42), (5.44) are sufficient to express h_0 in terms of $\tilde{g}_1$ and its derivatives. Note that the symmetry in eq. (5.43) gets fixed when one makes use of eq. (5.44).

Writing the action in terms of the new variables $\tilde{g}_1$, $\tilde{g}_2$ and using the Polyakov-Wiegmann formula, we obtain

$$S(g_1, g_2, k) \;\; \to \;\; S(\tilde{g}_1, \tilde{g}_2, k) = S_{WZNW}(\tilde{g}_2, k) \; + \; S_{WZNW}(\tilde{g}_1, k) \; + \; S_{WZNW}(h_0, k)$$

$$(5.48)$$

$$- \; \frac{k^2}{\pi} \int d^2 z \text{Tr} \tilde{g}_1 \partial \tilde{g}_1 \; S \; \bar{\partial} h_0 h_0^{-1},$$

where h_0 is a nonlocal function of $\tilde{g}_1$ satisfying (5.32), (5.34). Remarkably, after this change of variables, the field $\tilde{g}_2$ completely decouples from $\tilde{g}_1$.

The price we pay for the factorization is a highly nonlocal theory for the variable $\tilde{g}_1$. While $\tilde{g}_2$ is governed simply by a WZNW action, the action for $\tilde{g}_1$ is

$$S(\tilde{g}_1) \;\; = \;\; S_{WZNW}(\tilde{g}_1, k) \; + \; S_{WZNW}(h_0(\tilde{g}_1), k)$$

$$(5.49)$$

$$- \; \frac{k^2}{\pi} \int d^2 z \text{Tr}^2 \tilde{g}_1^{-1} \partial \tilde{g}_1 \; S \; \bar{\partial} h_0(\tilde{g}_1) h_0^{-1}(\tilde{g}_1),$$

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which is non-local as h_0 is a non-local function of $\tilde{g}_1$. In the Wess-Zumino term in $S_{WZNW}(h_0(\tilde{g}_1), k)$, h_0 and $\tilde{g}_1$ are extended to functions of z, $\bar{z}$ and an extra coordinate t and an equation analogous to (5.44) can be found for $\partial_t h_0 h_0^{-1}$. Then h_0 appears in (5.49) only through its derivatives, so that (5.49) can be written in terms of $\tilde{g}_1$ using (5.42), (5.44) and the analogous equation for $\partial_t h_0 h_0^{-1}$. The resulting action takes the following form

$$
\begin{aligned}
S(\tilde{g}_1) \;=\;& S_{WZNW}(\tilde{g}_1, k) \;+\; \frac{k^3}{\pi} \int d^2 z \operatorname{Tr}^2(\operatorname{Tr} S \tilde{g}_1^{-1} \partial \tilde{g}_1 \; \operatorname{Tr} S \tilde{g}_1^{-1} \bar{\partial} \tilde{g}_1) \\
&+\; \frac{k}{\pi} \int d^2 z \bar{\partial}(\tilde{g}_1^{-1} \partial \tilde{g}_1) \Psi \;+\; \mathcal{O}(S^3),
\end{aligned}
\tag{5.50}
$$

where

$$
\Psi(z, \bar{z}) = k^2 \int d^2 y G(z, \bar{z}; y, \bar{y}) \operatorname{Tr} S \; v(y, \bar{y}).
\tag{5.51}
$$

The important point to be made is that all interaction terms in eq. (5.50) are to be understood as normal ordered, so that all divergences coming from non-local terms do not occur.

For our purposes it is necessary to establish the duality symmetry of the non-local theory described by eq. (5.49). To this end, let us consider the following partition function

$$
Z_B(k, S) = \int \mathcal{D}\tilde{g}_1 \exp[-S(\tilde{g}_1)].
\tag{5.52}
$$

This emerges in the partition function of the system of interacting WZNW models

$$
Z(k, k, S) = Z_{WZNW}(k) Z_B(k, S).
\tag{5.53}
$$

Correspondingly formula (4.38) can be rewritten in terms of Z_B as follows

$$
Z_B(k, S) = J^{-1} \, Z_{WZNW}(k) \, \bar{Z}_{ghost} \, Z_{ghost} \, Z_{WZNW}(-k - 2c_V(G)) \, Z_B(-k - 2c_V(G), \tilde{S}),
\tag{5.54}
$$

where $\tilde{S}$ is the dual coupling constant matrix

$$
\tilde{S} = \frac{-S^{-1}}{4(k + 2c_V)k}.
\tag{5.55}
$$

Taking into account observations (4.29) and (4.32) we find the following identity

$$
Z_B(k, S) = Z_B(-k - 2c_V, \tilde{S}).
\tag{5.56}
$$

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This is the relation which manifests the precise equivalence between the two theories.

Apparently, when $S \to \infty$, $\tilde{S} \to 0$ and vice versa. Thus, equality (5.56) allows us to link the strong-coupling regime of one theory to the weak-coupling phase of another. In other words, we have got the opportunity to explore the quantum field theory with the large coupling constant. Certainly, the dual model is very non-trivial. Nevertheless, it can be studied at least within perturbation theory [8].

It is worth reminding the method of studying the dual theory [8]. To start with, we would like to point out some general aspects of this theory in the small $\tilde{S}$ limit. In this limit, the system of two interacting WZNW models can be regarded as a sum of two non-unitary WZNW models perturbed by the marginal (but not truly marginal) operator

$$X = J_2 \otimes \bar{J}_1, \tag{5.57}$$

where

$$J_2 = \frac{(k + 2c_V)}{2} h_2^{-1} \partial h_2, \qquad \bar{J}_1 = \frac{(k + 2c_V)}{2} \bar{\partial} h_1 h_1^{-1}. \tag{5.58}$$

Since X is not a truly marginal operator, the beta function of $\tilde{S}$ does not vanish for all $\tilde{S}$. At the same time, because X is marginal, the beta function starts with $\mathcal{O}(\tilde{S}^2)$ terms,

$$\beta = \mathcal{O}(\tilde{S}^2) \ + \ ... \tag{5.59}$$

Therefore, in order to check, for example, whether there are non-trivial fixed points or not, one has to go in computations to order $\tilde{S}^3$. Technically such computations seem to be quite involved as one has to expand to X^3. Fortunately, there is a trick which will allow us to overcome these technical obstacles. We shall show that the factorization simplifies the calculation of the beta function.

We proceed to make it clear that the limit $S \to \infty$ or $\tilde{S} \to 0$ does exist, and we, indeed, can perform perturbation around $\tilde{S} = 0$. This is important step in our discussion. According to identity (4.38) the limit $S \to \infty$ corresponds to the CFT with the Virasoro central charge

$$c_\infty = 2 \dim G. \tag{5.60}$$

This formula comes from the right hand side of eq. (4.38). On the left hand side of eq.

(4.38), the limit $S \to \infty$ gives rise effectively to the theory

$$S(S \to \infty) = -\frac{k^2}{\pi} \int d^2 z \mathrm{Tr}^2(g_1^{-1}\partial g_1 \, S \, \bar{\partial}g_2 g_2^{-1}). \tag{5.61}$$

Let us assume that $S = \rho \, I$, where $\rho \to \infty$. Then, eq. (5.61) takes the form

$$S(\rho) = -\frac{k^2\rho}{\pi} \int d^2 z \mathrm{Tr}(g_1^{-1}\partial g_1 \bar{\partial}g_2 g_2^{-1}). \tag{5.62}$$

The latter can be regarded as the non-linear sigma model

$$S(\lambda) = -\frac{1}{4\pi\lambda^2} \int d^2 z \mathrm{Tr}(g_1^{-1}\partial g_1 \bar{\partial}g_2 g_2^{-1}) \tag{5.63}$$

taken in the limit $\lambda^2 = 1/(4k^2\rho) \to 0$. In the given limit, the non-linear sigma model is equivalent to the CFT of the direct sum of two Abelian affine current algebras with dimensions equal to $\dim G$. The Virasoro central charge of the given CFT is equal to $2 \dim G$. Thus, we have shown that the limit $S \to \infty$ leads us to the same CFT as the limit $\tilde{S} \to 0$. Therefore, this agreement justifies the existence of the both limits and their consistency to each other. This is a crucial observation for pursuing expansion around the CFT at $\tilde{S} = 0$.

For the dual theory the factorized one is given by

$$\begin{aligned} \tilde{S}(\tilde{h}_1) &= S_{WZNW}(\tilde{h}_1, -k - 2c_V) - \frac{(k + 2c_V)^3}{\pi} \int d^2 z \mathrm{Tr}^2(\mathrm{Tr}\tilde{S}\tilde{h}_1^{-1}\partial\tilde{h}_1 \, \mathrm{Tr}\tilde{S}\tilde{h}_1^{-1}\bar{\partial}\tilde{h}_1) \\ &\qquad (5.64) \\ &\quad - \frac{(k + 2c_V)}{\pi} \int d^2 z \bar{\partial}(\tilde{h}_1^{-1}\partial\tilde{h}_1)\tilde{\Psi} \, + \, \mathcal{O}(\tilde{S}^3). \end{aligned}$$

The given non-local theory can be properly understood as follows. We consider the non-local term and all other higher in $\tilde{S}$ terms as being normal ordered with respect to the two first terms in eq. (5.64). Whereas the second term in eq. (5.64) is thought of as being a perturbation on the conformal WZNW model at level $-k - 2c_V$. Such a view of the given non-local functional allows us to develop selfconsistent expansion in the small coupling $\tilde{S}$.

The equation of motion of the WZNW model is as follows

$$\bar{\partial}(\tilde{h}_1^{-1}\partial\tilde{h}_1) = 0. \tag{5.65}$$

When the second term in eq. (5.64) is turned on, the equation of motion gets perturbed according to

$$\bar{\partial}(\tilde{h}_1^{-1}\partial\tilde{h}_1) = \mathcal{O}(\tilde{S}), \tag{5.66}$$

where the right hand side of eq. (5.66) is expressed in terms of normal ordered products of conformal operators of the CFT. Eq. (5.66) can be used further for normal ordering the non-local term and higher order terms in eq. (5.64). Now it becomes clear that since the non-local term contains $\bar{\partial}(\tilde{h}_1^{-1}\partial\tilde{h}_1)$, its normal ordered expression will be of order $\tilde{S}^3$ due to the equation of motion (5.66). Therefore, if we confine ourselves to leading in $\tilde{S}$ orders, we can stick to the following approximation

$$\tilde{S}(\tilde{h}_1) = S_{WZNW}(\tilde{h}_1, -k - 2c_V) - \frac{(k + 2c_V)^3}{\pi} \int d^2z \mathrm{Tr}(\mathrm{Tr}\tilde{S}\tilde{h}_1^{-1}\partial\tilde{h}_1 \; \mathrm{Tr}\tilde{S}\tilde{h}_1^{-1}\bar{\partial}\tilde{h}_1) + \mathcal{O}(\tilde{S}^3). \tag{5.67}$$

Remarkably, the given approximation is a local theory.

Now we want to exhibit that in the space of models described by eq. (5.67) there is a class of renormalizable theories with the coupling constant matrix $\tilde{S}$ having the following structure

$$\tilde{S} = \sigma \; \hat{S}, \tag{5.68}$$

with σ being a small parameter and $\hat{S}$ a fixed matrix which is not subject to renormalization. Eq. (5.68) can be understood as a line in the space of coupling constants.

With the coupling as in eq. (5.68) the theory in eq. (5.67) reads off

$$S(\tilde{h}_1) = S_{WZNW}(\tilde{h}_1, -k - 2c_V) - \epsilon \int d^2z \; O^{\hat{S}}(z, \bar{z}) + \mathcal{O}(\sigma^3), \tag{5.69}$$

where we have introduced the following notations

$$\epsilon = \frac{4(k + 2c_V)}{\pi}\sigma^2, \qquad O^{\hat{S}} = \hat{S}^{a\bar{a}}\hat{S}^{b\bar{b}} : \tilde{J}^a \bar{\tilde{J}}^b \tilde{\phi}^{b\bar{b}} : . \tag{5.70}$$

Here

$$\begin{aligned}
\tilde{J} &= \frac{(k + 2c_V)}{2}\tilde{h}_1^{-1}\partial\tilde{h}_1, \\
\bar{\tilde{J}} &= \frac{(k + 2c_V)}{2}\bar{\partial}\tilde{h}_1\tilde{h}_1^{-1}, \\
\tilde{\phi}^{a\bar{a}} &= \mathrm{Tr}(\tilde{h}_1^{-1}t^a\tilde{h}_1t^{\bar{a}}).
\end{aligned} \tag{5.71}$$

The operator $O^{\hat{S}}$ in eq. (5.69) is a Virasoro primary operator with the conformal weight

$$\Delta_O = 1 - \frac{c_V}{k + c_V}. \tag{5.72}$$

It is transparent that Δ_O lies in the interval between 0 and 1 and, thus, it is a relevant operator in the non-unitary WZNW model. Despite the non-unitarity of the WZNW theory, $O^{\hat{S}}$ belongs to the unitary range of the Kac-Kazhdan determinant and, hence, it provides a unitary representation of the Virasoro algebra.

In order for the theory in eq. (5.69) to be renormalizable, the operator $O^{\hat{S}}$ has to form a closed OPE algebra. The last condition results in an algebraic equation for the matrix $\hat{S}$ [12]. Solutions to this equation yield renormalizable relevant perturbations on the WZNW model.

We shall show that the limit $\tilde{S} \to 0$ is consistent with the limit $k \to \infty$. Therefore, for our goal it will be sufficient to consider the large k solutions for $\hat{S}$. In this limit, there is one invertible solution given by [12]

$$\hat{S}^{a\bar{a}}\hat{S}^{b\bar{a}} = \frac{\delta^{ab}}{c_V} + ..., \tag{5.73}$$

where dots stand for the higher in $1/k$ corrections. With the given $\hat{S}$ the theory is renormalizable and the beta function can be computed:

$$\beta(\epsilon) = (2 - 2\Delta_O)\epsilon - \pi \epsilon^2 + ... \tag{5.74}$$

Note that as ϵ is proportional to σ^2, eq. (5.74) agrees with eq. (5.59).

Now it becomes clear that the beta function in eq. (5.74) has a non-trivial conformal point

$$\epsilon^* = \frac{4}{\pi k} + ... \tag{5.75}$$

Recalling the relation between ϵ and $\tilde{S}$ given by eq. (5.70) one can see that the large k limit is consistent with the limit $\tilde{S} \to 0$. Furthermore, because of this relation, solution (5.75) gives rise to two solutions for $\tilde{S}$. However, only one of these two critical points can be reached along the renormalization group flow [8]. Indeed, going to the point $\tilde{S}^* > 0$ necessitates the flow to pass the Polyakov-Wiegmann critical point, at which the system acquires the gauge symmetry. Therefore, perturbation in the direction to $\tilde{S}^* > 0$ cannot

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be continuous. At the same time, it might be likely the case that these two critical points are related by some duality symmetry [1]. The CFT's corresponding to these two conformal points must be essentially equivalent and, hence, we can carry on with one of these points. From now on we shall study the fixed point $\tilde{S}^* < 0$.

Known ϵ^* we find the critical point $\tilde{S}^* < 0$ and, correspondingly, a conformal point for the system of two interacting WZNW models

$$S^* = \frac{1}{2\sqrt{2}k} + \; ... \tag{5.76}$$

This point does not coincide with the PW conformal point and is new.

Although, in the limit $k \to \infty$, the critical point $S^* \to 0$, it cannot be seen by expanding the beta function around $S = 0$. In fact, the given conformal point becomes visible only in the strong-coupling phase via expansion around $S = \infty$. The strong-coupling region is completely covered by the weak-coupling phase of the dual theory. Together, the weak-coupling phase of the system of two interacting WZNW models and the weak-coupling phase of the dual theory, cover the whole range of positive values of the coupling constant s in eq. (3.18).

Note that at $S = \infty$ there is another fixed point with the Virasoro central charge given by eq. (5.60).

We can say more about the point S^* in eq. (5.76) noticing that the perturbative CFT (5.69) at this point can be identified with an exact CFT [13]. The latter is nothing but the WZNW model with level k,

$$S(\tilde{h}_1, S = S^*) = S_{WZNW}(\tilde{h}_1, k). \tag{5.77}$$

By using this result, one can derive from eq. (5.54) the following expression

$$Z(k, k, S^*) = Z^2_{WZNW}(k). \tag{5.78}$$

This result is rather unexpected. It exhibits that in the space of interacting WZNW models there exists a point at which the interacting theory becomes entirely equivalent to the system of non-interacting WZNW models. In other words, the theory with the interaction suddenly becomes free again at the particular value of the coupling. The

similar effect has been described for the system of spinor fields with the four-fermionic interaction [11]. In fact, the fixed point S^* is related to the so-called isoscalar Dashen-Frishman conformal point of the non-Abelian Thirring model [8].

Due to the equivalence between the system of two interacting WZNW models and the PCM described in section 3, the weak-coupling phase of the later is related to the strong-coupling regime of the former. However, at the critical point we just have found above, the given equivalence has to break down, like at the point $S = 0$ or at the Polyakov-Wiegmann conformal point. Exploring of the PCM in the vicinity of the Dashen-Frishman fixed point might be an interesting task.

In this section we analyzed the conformal property of the system of two interacting WZNW models in the strong-coupling phase. Also we pointed out that renormalizability of this theory admits many other $\hat{S}$'s to exist, that will lead to new critical points. In the next section we shall discuss the algebraic structure of the theory at hand to reveal the possibility of existence of many more non-perturbative conformal points.

6   The current-current algebraic approach

Based on the results obtained in the previous sections we want to propose a non-perturbative way of finding conformal points of the system of two interacting WZNW models.

We have exhibited in the previous section that one critical point can be found in the large k limit due to the duality. However, there may exist some other fixed points at particular values of k. Certainly if such critical points actually exist, they cannot be found in the large k limit. In order to discover these points, one should solve the theory exactly that is, of course, an extremely difficult task. However, we shall show that the large k limit provides insight into the algebraic structure of the theory, which turns out to be a key point in the search for non-perturbative conformal points.

According to definition, at a conformal point the system is characterized by the stress-tensor with the two analytic components T and $\bar{T}$,

$$\bar{\partial}T = 0, \qquad \partial\bar{T} = 0. \tag{6.79}$$

The third component which is equal to the trace of the stress-tensor vanishes due to

the conformal symmetry. In addition, T and $\bar{T}$ form two copies of the Virasoro algebra. Thus, whenever the theory possesses the conformal symmetry, it must have two analytic operators forming two Virasoro algebras.

At the classical level, the system of two interacting WZNW models is conformal. So is the non-local theory described by eq. (5.49) as well as its dual one. Let us consider the dual theory. The classical stress-tensor has the following two components

$$T(\tilde{S}) \;=\; \frac{(-1)}{k+2c_V}\tilde{J}^2 \;+\; \frac{(-1)}{k+2c_V}\tilde{J}^2_{\tilde{h}_0} \;+\; 4\tilde{J}(\mathrm{Tr}\tilde{S}\tilde{J}_{\tilde{h}_0}),$$

$$\bar{T}(\tilde{S}) \;=\; \frac{(-1)}{k+2c_V}\bar{\tilde{J}}^2 \;+\; \frac{(-1)}{k+2c_V}\bar{\tilde{J}}^2_{\tilde{h}_0} \;+\; 4\bar{\tilde{J}}(\mathrm{Tr}\tilde{S}\bar{\tilde{J}}_{\tilde{h}_0}),$$

(6.80)

where

$$\tilde{J} \;=\; \frac{(k+2c_V)}{2}\tilde{h}_1^{-1}\partial\tilde{h}_1,$$

$$\bar{\tilde{J}} \;=\; \frac{(k+2c_V)}{2}\bar{\partial}\tilde{h}_1\tilde{h}_1^{-1},$$

$$\partial\tilde{h}_0\tilde{h}_0^{-1} \;=\; 2(k+2c_V)\mathrm{Tr}\tilde{S}\tilde{h}_1^{-1}\partial\tilde{h}_1.$$

(6.81)

Using the equation for $\tilde{h}_0$, the holomorphic component T can be altered to the local form in terms of $\tilde{h}_1$:

$$T(\tilde{S}) = \left(\frac{-\delta^{ab}}{k+2c_V} \;+\; 4(k+2c_V)\tilde{S}^{a\bar{a}}\tilde{S}^{b\bar{a}} \right) \tilde{J}^a\tilde{J}^b.$$

(6.82)

This is curious, because

$$\bar{\partial}\tilde{J} \neq 0,$$

(6.83)

whereas

$$\bar{\partial}T(\tilde{S}) = 0.$$

(6.84)

In other words, the non-analytic part of the function $\tilde{J}(z,\bar{z})$ does not contribute into the holomorphic component of the classical stress-tensor. Therefore, if we take $\tilde{J}(z,\bar{z})$ at the origin, i.e. at $z = \bar{z} = 0$, then there will be no evolution in the $\bar{z}$-direction for the quantity $T(\tilde{S})$.

At $z = \bar{z} = 0$, we have

$$T(0) = \left(\frac{-\delta^{ab}}{k+2c_V} \;+\; 4(k+2c_V)\tilde{S}^{a\bar{a}}\tilde{S}^{b\bar{a}} \right) \tilde{J}^a(0,0)\tilde{J}^b(0,0).$$

(6.85)

At all other moments of time, we must have

$$T(z) = \left(\frac{-\delta^{ab}}{k + 2c_V} + 4(k + 2c_V)\tilde{S}^{a\bar{a}}\tilde{S}^{b\bar{a}} \right) \hat{J}^a(z)\hat{J}^b(z), \qquad (6.86)$$

where

$$\hat{J}^a(z) \equiv \tilde{J}^a(z, 0). \qquad (6.87)$$

Let us assume that $\tilde{J}^a(z, \bar{z})$ form the current algebra with respect to the Poisson brackets

$$\{\tilde{J}^a(z, \bar{z}), \ \tilde{J}^b(y, \bar{z})\} = f_c^{ab}\tilde{J}^c(z, \bar{z})\delta(z, y) + \ central \ term. \qquad (6.88)$$

At the classical level $\tilde{J}$ transforms as a primary field under the conformal group. Correspondingly, the algebra in eq. (6.88) is invariant under evolution in the $\bar{z}$-direction. Therefore, it will hold for $\hat{J}(z)$ as well,

$$\{\hat{J}^a(z), \ \hat{J}^b(y)\} = f_c^{ab}\hat{J}^c(z)\delta(z, y) + \ central \ term. \qquad (6.89)$$

Unfortunately, algebra (6.88) is inconsistent with the construction in eq. (6.82) for arbitrary $\tilde{S}$. It has been proven in [14] that the current algebra can be consistent with the Virasoro algebra only at some special values of the matrix L_{ab} in the construction

$$T = L_{ab} \ \hat{J}^a\hat{J}^b. \qquad (6.90)$$

Whereas in the case under consideration, $T(\tilde{S})$ forms the Virasoro algebra with arbitrary $\tilde{S}$. Hence, at the classical level, the quantity $\hat{J}$ has to have a more complicated algebra which has to be consistent with the classical Virasoro algebra for arbitrary $\tilde{S}$. In this context, it might be interesting to know, whether the algebra found in [15] is the algebra forming by $\hat{J}$.

The situation may change in quantum theory. Indeed, the classical stress-tensor, in the course of quantization, is subject to renormalization even at conformal points [11],[4]. Therefore, the quantum consistency condition between the current algebra and the Virasoro algebra can be different from the classical consistency condition due to renormalization.

In order to see how the renormalization works, we turn to the conformal point in the large k limit. It has been shown in the previous section that the dual non-local theory in

eq. (5.64) flows to the WZNW model with level k,

$$\tilde{S}(\tilde{S}^*) = S_{WZNW}(k). \tag{6.91}$$

In other words, the stress-tensor in eqs. (6.80) takes the affine-Sugawara form at the given conformal point,

$$\bar{T}(\tilde{S}^*) = \frac{1}{k + c_V} : \bar{\tilde{\mathcal{J}}}^2 :, \tag{6.92}$$

where $\bar{\tilde{\mathcal{J}}}$ is the affine current of the WZNW model with level k. The last expression exhibits that the classical stress-tensor $T(\tilde{S})$ gets renormalized to the Sugawara stress-tensor at the point $\tilde{S}^*$. The current $\bar{\tilde{\mathcal{J}}}$ coincides with the renormalized antiholomorphic current

$$\bar{\tilde{\mathcal{J}}} = \bar{\tilde{J}} - 2(k + 2c_V)\tilde{h}_1(\text{Tr}\tilde{S}\bar{\tilde{J}}_{\tilde{h}_0})\tilde{h}_1^{-1}. \tag{6.93}$$

At the classical level this current forms the affine algebra with level $-k - 2c_V$. In the course of quantization $\bar{\tilde{\mathcal{J}}}$ continues to be analytic due to the equation of motion. However, it undergoes some renormalization which affects the central charge. It turns out that in the large k limit it is not hard to compute the renormalized central charge of the renormalized affine current. By using perturbation theory we find

$$k_R = -k + 2k^3\sigma^2 + \dots \tag{6.94}$$

At the critical point S^*, the renormalized central charge k_R is given by

$$k_R(S^*) = -k + 2k + \dots = k, \tag{6.95}$$

where dots stand for higher in $1/k$ corrections. Thus, the renormalized current forms the affine algebra at level k (at least in the large k limit).

Finally we arrive at the conclusion that the classical stress-tensor in eq. (6.80) at the conformal point S^* gets renormalized to the operator

$$T(\tilde{S}^*) = \frac{1}{k + c_V} : \tilde{\mathcal{J}}^2 :, \tag{6.96}$$

with $\tilde{\mathcal{J}}$ the renormalized classical affine current. In other words, the quantity $\hat{J}$ goes to $\tilde{\mathcal{J}}$ at the conformal point S^*. One can check that

$$\frac{-\delta^{ab}}{k + 2c_V} + 4(k + 2c_V)\tilde{S}^{a\bar{a}}\tilde{S}^{b\bar{a}} \xrightarrow{\tilde{S} \to \tilde{S}^*} \frac{\delta^{ab}}{k} + \dots \tag{6.97}$$

Therefore,

$$\hat{J} \xrightarrow{\tilde{S} \to \tilde{S}^*} \tilde{\mathcal{J}}. \tag{6.98}$$

Now let us return to the theory in eq. (5.49). In the latter, the classical affine current forms the algebra with level k. Hence, the given classical current is not needed to undergo any renormalization to emerge in the Sugawara construction. This is also true because the large k limit is the strong coupling limit for the given theory. The renormalization affects only the coefficient matrix in the stress-tensor:

$$T(L) = L_{ab} : \mathcal{J}^a \mathcal{J}^b :, \tag{6.99}$$

where $\mathcal{J}$ is the affine current with level k, whereas L_{ab} is a matrix related to the coupling constant matrix $S^{a\bar{a}}$. The given construction does not contradict to the renormalizability of the classical stress-tensor. Indeed, for the antiholomorphic component of the conjectured stress-tensor, one obtains

$$\bar{T}(L) = L \left[\bar{\tilde{J}}^2 \ + \ 4k\bar{\tilde{J}}\tilde{g}_1 (\mathrm{Tr}\tilde{S}\bar{\tilde{J}}_{\tilde{h}_0})\tilde{g}_1^{-1} \ + \ 4k^2 (\mathrm{Tr}\tilde{S}\bar{\tilde{J}}_{\tilde{h}_0})^2 \right]. \tag{6.100}$$

One can see that the expression in eq. (6.100) has the same terms as the classical $\bar{T}(S)$ only with different coefficients. Therefore, the renormalizability allows us to consider the operator in eq. (6.100) as the quantum renormalized component of the energy stress-tensor at conformal points. The condition of conformal invariance amounts to imposing the Virasoro algebra on $\bar{T}(L)$. This fixes values of the matrix L [14]. However, this condition does not tell us how the matrix L is related to the coupling constant matrix S. This relation has to be derived from dynamics. In the large k limit, one can find

$$L = -\frac{I}{k} \ + \ \frac{2c_V}{k}\hat{S}^2 \ + \ ... \tag{6.101}$$

The left hand side of eq. (6.101) can be found from the master-Virasoro equation which describes embeddings of the Virasoro algebra into the affine algebra [14]. As a result, the system of two interacting WZNW models may be used for Lagrangian interpretation of the algebraic affine-Virasoro construction.

All in all we have established that equation (6.101) (or its non-perturbative generalization) represents a sufficient condition of the conformal invariance of the system of

two interacting WZNW models. We have checked in [1] that at $k = 4$ the system of two interacting $SL(2)$ WZNW models is conformal with a continuous coupling constant. Similarly, the affine-Virasoro construction on $SL(2)$ also has a continuous solution at $k = 4$ [16]. This is a non-trivial evidence that relation (6.101) is also a necessary condition of the conformal symmetry. Some non-perturbative arguments in favour of it have been presented in [17] as well.

7   Conclusion

We have exhibited that the system of two interacting WZNW models has the weak coupling phase and the strong coupling phase. The scaling properties of the theory in these two phases are completely different. We have presented both perturbative and non-perturbative arguments that in the strong coupling phase the theory flows to non-trivial conformal points, which may correspond to some of the string compactifications. The hope is that the given properties of interacting WZNW models will turn out to be useful for understanding the string theory symmetries.

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